

8. Legendre Functions

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Contents

	Page
Mathematical Properties	332
Notation	332
8.1. Differential Equation	332
8.2. Relations Between Legendre Functions	333
8.3. Values on the Cut	333
8.4. Explicit Expressions	333
8.5. Recurrence Relations	333
8.6. Special Values	334
8.7. Trigonometric Expansions	335
8.8. Integral Representations	335
8.9. Summation Formulas	335
8.10. Asymptotic Expansions	335
8.11. Toroidal Functions	336
8.12. Conical Functions	337
8.13. Relation to Elliptic Integrals	337
8.14. Integrals	337
Numerical Methods	339
8.15. Use and Extension of the Tables	339
References	340
Table 8.1. Legendre Function-First Kind $P_n(x)$ ($x \leq 1$)	342
$x=0(.01)1, n=0(1)3, 9, 10, 5-8D$	
Table 8.2. Derivative of the Legendre Function-First Kind $P'_n(x)$ ($x \leq 1$)	344
$x=0(.01)1, n=1(1)4, 9, 10, 5-7D$	
Table 8.3. Legendre Function-Second Kind $Q_n(x)$ ($x \leq 1$)	346
$x=0(.01)1, n=0(1)3, 9, 10, 8D$	
Table 8.4. Derivative of the Legendre Function-Second Kind $Q'_n(x)$ ($x \leq 1$)	348
$x=0(.01)1, n=0(1)3, 9, 10, 6-8D$	
Table 8.5. Legendre Function-First Kind $P_n(x)$ ($x \geq 1$)	350
$x=1(.2)10, n=0(1)5, 9, 10, \text{ exact or } 6S$	
Table 8.6. Derivative of the Legendre Function-First Kind $P'_n(x)$ ($x \geq 1$)	351
$x=1(.2)10, n=1(1)5, 9, 10, 6S$	
Table 8.7. Legendre Function-Second Kind $Q_n(x)$ ($x \geq 1$)	352
$x=1(.2)10, n=0(1)3, 9, 10, 6S$	
Table 8.8. Derivative of the Legendre Function-Second Kind $Q'_n(x)$ ($x \geq 1$)	353
$x=1(.2)10, n=0(1)3, 9, 10, 6S$	

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¹ National Bureau of Standards.

8. Legendre Functions

Mathematical Properties

Notation

The conventions used are $z=x+iy$, x, y real, and in particular, x always means a real number in the interval $-1 \leq x \leq +1$ with $\cos \theta = x$ where θ is likewise a real number; n and m are positive integers or zero; ν and μ are unrestricted except where otherwise indicated.

Other notations are:

$$P_n(x) \text{ for } \frac{n!P_n(x)}{(2n-1)!!}$$

$$P_{nm}(x) \text{ for } (-1)^m P_n^m(x)$$

$$T_n^m(x) \text{ for } (-1)^m P_n^m(x)$$

$$\overline{P}_n^m(x) \text{ for } (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_n^m(x)$$

$$\mathfrak{P}_\nu^\mu(z) \text{ for } P_\nu^\mu(z), \mathfrak{Q}_\nu^\mu(z) \text{ for } Q_\nu^\mu(z) \quad (\Re z > 1)$$

$$\mathfrak{Q}_\nu^\mu(z) \text{ for } e^{i\pi\mu} Q_\nu^\mu(z)$$

$$Q_\nu^\mu(z) \text{ for } \frac{\sin(\nu+\mu)\pi}{\sin \nu\pi} Q_\nu^\mu(z)$$

Various other definitions of the functions occur as well as mixing of definitions.

8.1. Differential Equation

8.1.1

$$(1-z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}] w = 0$$

Solutions

(Degree ν and order μ with singularities at $z = \pm 1, \infty$ as ordinary branch points— μ, ν arbitrary complex constants.)

$P_\nu^\mu(z), Q_\nu^\mu(z)$ —Associated Legendre Functions (Spherical Harmonics) of the First and Second Kinds²

$$|\arg(z \pm 1)| < \pi, \quad |\arg z| < \pi$$

$$(z^2 - 1)^{\frac{1}{2}\mu} = (z-1)^{\frac{1}{2}\mu} (z+1)^{\frac{1}{2}\mu}$$

(For $P_\nu^\mu(z)$, $\mu=0$, Legendre polynomials, see chapter 22.)

8.1.2

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1} \right]^{\frac{1}{2}\mu} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \quad (|1-z| < 2)$$

(For $F(a, b; c; z)$ see chapter 15.)

$$8.1.3 \quad Q_\nu^\mu(z) = e^{i\mu\pi} 2^{-\nu-1} \pi^{\frac{1}{2}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1} (z^2-1)^{\frac{1}{2}\mu} F\left(1+\frac{\nu}{2}+\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \quad (|z| > 1)$$

Alternate Forms

(Additional forms may be obtained by means of the transformation formulas of the hypergeometric function, see [8.1].)

$$8.1.4 \quad P_\nu^\mu(z) = 2^\mu \pi^{\frac{1}{2}} (z^2-1)^{-\frac{1}{2}\mu} \left\{ \frac{F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right) \Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)} - 2z \frac{F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right) \Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}\right)} \right\} \quad (|z^2| < 1)$$

$$8.1.5 \quad P_\nu^\mu(z) = \frac{2^{-\nu-1} \pi^{-\frac{1}{2}} \Gamma(-\frac{1}{2}-\nu) z^{-\nu+\mu-1}}{(z^2-1)^{\mu/2} \Gamma(-\nu-\mu)} F\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \nu+\frac{3}{2}; z^{-2}\right) + \frac{2^\nu \Gamma(\frac{1}{2}+\nu) z^{\nu+\mu}}{(z^2-1)^{\mu/2} \Gamma(1+\nu-\mu)} F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}-\nu; z^{-2}\right) \quad (|z^{-2}| < 1)$$

$$8.1.6 \quad e^{-i\mu\pi} Q_\nu^\mu(z) = \frac{\Gamma(1+\nu+\mu) \Gamma(-\mu) (z-1)^{\frac{1}{2}\mu} (z+1)^{-\frac{1}{2}\mu}}{2\Gamma(1+\nu-\mu)} F\left(-\nu, 1+\nu; 1+\mu; \frac{1-z}{2}\right) + \frac{1}{2} \Gamma(\mu) (z+1)^{\frac{1}{2}\mu} (z-1)^{-\frac{1}{2}\mu} F\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right) \quad (|1-z| < 2) \quad *$$

² The functions $Y_n^m(\theta, \varphi) = \frac{\cos m\varphi}{\sin m\varphi} P_n^m(\cos \theta)$ called surface harmonics of the first kind, tesseral for $m < n$ and sectoral for $m = n$. With $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, they are everywhere one valued and continuous functions on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$ where $x = \sin \theta \cos \varphi$, $y = \sin \theta \sin \varphi$ and $z = \cos \theta$.

*See page II.

$$8.1.7 \quad e^{-i\mu\pi} Q_{\nu}^{\mu}(z) = \pi^{\frac{1}{2}} 2^{\mu} (z^2 - 1)^{-\frac{1}{2}\mu} \left\{ \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)}{2\Gamma\left(1 + \frac{\nu}{2} - \frac{\mu}{2}\right)} e^{\pm i\frac{1}{2}\pi(\mu - \nu - 1)} F\left(-\frac{\nu}{2} - \frac{\mu}{2}, \frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}, \frac{1}{2}; z^2\right) \right. \\ \left. + \frac{z\Gamma\left(1 + \frac{\nu}{2} + \frac{\mu}{2}\right) e^{\pm i\frac{1}{2}\pi(\mu - \nu)}}{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}\right)} F\left(\frac{1}{2} - \frac{\nu}{2} - \frac{\mu}{2}, 1 + \frac{\nu}{2} - \frac{\mu}{2}, \frac{3}{2}; z^2\right) \right\} \quad (|z^2| < 1)$$

Wronskian

(Upper and lower signs according as $\mathcal{R}z \geq 0$.)

8.1.8

$$W\{P_{\nu}^{\mu}(z), Q_{\nu}^{\mu}(z)\} = \frac{e^{i\mu\pi} 2^{2\mu} \Gamma\left(\frac{\nu + \mu + 2}{2}\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{(1 - z^2) \Gamma\left(\frac{\nu - \mu + 2}{2}\right) \Gamma\left(\frac{\nu - \mu + 1}{2}\right)}$$

$$8.1.9 \quad W\{P_{\nu}(z), Q_{\nu}(z)\} = -(z^2 - 1)^{-1}$$

8.2. Relations Between Legendre Functions

Negative Degree

$$8.2.1 \quad P_{-\nu-1}^{\mu}(z) = P_{\nu}^{\mu}(z)$$

8.2.2

$$Q_{-\nu-1}^{\mu}(z) = \{-\pi e^{i\mu\pi} \cos \nu\pi P_{\nu}^{\mu}(z) + Q_{\nu}^{\mu}(z) \sin [\pi(\nu + \mu)]\} / \sin [\pi(\nu - \mu)]$$

Negative Argument ($\mathcal{R}z \geq 0$)

8.2.3

$$P_{\nu}^{\mu}(-z) = e^{\mp i\mu\pi} P_{\nu}^{\mu}(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin [\pi(\nu + \mu)] Q_{\nu}^{\mu}(z)$$

8.2.4

$$Q_{\nu}^{\mu}(-z) = -e^{\pm i\mu\pi} Q_{\nu}^{\mu}(z)$$

Negative Order

8.2.5

$$P_{\nu}^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[P_{\nu}^{\mu}(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin (\mu\pi) Q_{\nu}^{\mu}(z) \right]$$

8.2.6

$$Q_{\nu}^{-\mu}(z) = e^{-2i\mu\pi} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_{\nu}^{\mu}(z)$$

Degree $\mu + \frac{1}{2}$ and Order $\nu + \frac{1}{2}$

$\mathcal{R}z > 0$

$$8.2.7 \quad P_{-\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right) = \frac{(z^2-1)^{1/4} e^{-i\mu\pi} Q_{\nu}^{\mu}(z)}{(\frac{1}{2}\pi)^{1/2} \Gamma(\nu + \mu + 1)}$$

8.2.8

$$Q_{-\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right) = -i(\frac{1}{2}\pi)^{1/2} \Gamma(-\nu-\mu) (z^2-1)^{1/4} e^{-i\mu\pi} P_{\nu}^{\mu}(z)$$

8.3. Values on the Cut

$(-1 < x < 1)$

8.3.1

$$P_{\nu}^{\mu}(x) = \frac{1}{2} [e^{i\frac{1}{2}\mu\pi} P_{\nu}^{\mu}(x + i0) + e^{-i\frac{1}{2}\mu\pi} P_{\nu}^{\mu}(x - i0)]$$

8.3.2

$$P_{\nu}^{\mu}(x) = e^{\pm i\frac{1}{2}\mu\pi} P_{\nu}^{\mu}(x \pm i0) \quad *$$

8.3.3

$$= i\pi^{-1} e^{-i\mu\pi} [e^{-i\frac{1}{2}\mu\pi} Q_{\nu}^{\mu}(x + i0) - e^{i\frac{1}{2}\mu\pi} Q_{\nu}^{\mu}(x - i0)] \quad *$$

8.3.4

$$Q_{\nu}^{\mu}(x) = \frac{1}{2} e^{-i\mu\pi} [e^{-i\frac{1}{2}\mu\pi} Q_{\nu}^{\mu}(x + i0) + e^{i\frac{1}{2}\mu\pi} Q_{\nu}^{\mu}(x - i0)]$$

(Formulas for $P_{\nu}^{\mu}(x)$ and $Q_{\nu}^{\mu}(x)$ are obtained with the replacement of $z-1$ by $(1-x)e^{\pm i\pi}$, (z^2-1) by $(1-x^2)e^{\pm i\pi}$, $z+1$ by $x+1$ for $z = x \pm i0$.)

8.4. Explicit Expressions

$(x = \cos \theta)$

8.4.1

$$P_0(z) = 1 \quad P_0(x) = 1$$

8.4.2

$$Q_0(z) = \frac{1}{2} \ln \left(\frac{z+1}{z-1} \right) \quad Q_0(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \\ = xF\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right)$$

8.4.3

$$P_1(z) = z \quad P_1(x) = x = \cos \theta$$

8.4.4

$$Q_1(z) = \frac{z}{2} \ln \left(\frac{z+1}{z-1} \right) - 1 \quad Q_1(x) = \frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) - 1$$

8.4.5

$$P_2(z) = \frac{1}{2}(3z^2 - 1) \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \\ = \frac{1}{4}(3 \cos 2\theta + 1)$$

8.4.6

$$Q_2(z) = \frac{1}{2} P_2(z) \ln \left(\frac{z+1}{z-1} \right) - \frac{3z}{2} \quad Q_2(x) = \\ - \frac{3x}{2} \left(\frac{3x^2 - 1}{4} \right) \ln \left(\frac{1+x}{1-x} \right) - \frac{3x}{2}$$

8.5. Recurrence Relations

(Both P_{ν}^{μ} and Q_{ν}^{μ} satisfy the same recurrence relations.)

Varying Order

8.5.1

$$P_{\nu}^{\mu+1}(z) = (z^2 - 1)^{-\frac{1}{2}} \{ (\nu - \mu) z P_{\nu}^{\mu}(z) - (\nu + \mu) P_{\nu-1}^{\mu}(z) \}$$

8.5.2

$$(z^2-1) \frac{dP_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{\frac{1}{2}} P_{\nu-1}^{\mu-1}(z) - \mu z P_\nu^\mu(z)$$

Varying Degree

8.5.3

$$(\nu-\mu+1)P_{\nu+1}^\mu(z) = (2\nu+1)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)$$

$$8.5.4 \quad (z^2-1) \frac{dP_\nu^\mu(z)}{dz} = \nu z P_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)$$

Varying Order and Degree

$$8.5.5 \quad P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{\frac{1}{2}} P_\nu^{\mu-1}(z)$$

8.6. Special Values

$$x=0$$

8.6.1

$$P_\nu^\mu(0) = 2^{\mu-1} \pi^{-\frac{1}{2}} \cos \left[\frac{1}{2} \pi (\nu+\mu) \right] \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right) / \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + 1\right)$$

8.6.2

$$Q_\nu^\mu(0) = -2^{\mu-1} \pi^{-\frac{1}{2}} \sin \left[\frac{1}{2} \pi (\nu+\mu) \right] \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right) / \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + 1\right)$$

8.6.3

$$\left[\frac{dP_\nu^\mu(x)}{dx} \right]_{x=0} = 2^{\mu+1} \pi^{-\frac{1}{2}} \sin \left[\frac{1}{2} \pi (\nu+\mu) \right] \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + 1\right) / \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2}\right)$$

8.6.4

$$\left[\frac{dQ_\nu^\mu(x)}{dx} \right]_{x=0} = 2^\mu \pi^{\frac{1}{2}} \cos \left[\frac{1}{2} \pi (\nu+\mu) \right] \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + 1\right) / \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2}\right)$$

8.6.5

$$W\{P_\nu^\mu(x), Q_\nu^\mu(x)\}_{x=0} = \frac{2^{2\mu} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + 1\right) \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + 1\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2}\right)}$$

$$\mu = m = 1, 2, 3, \dots$$

8.6.6

$$P_\nu^m(z) = (z^2-1)^{\frac{1}{2}m} \frac{d^m P_\nu(z)}{dz^m},$$

$$P_\nu^m(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m P_\nu(x)}{dx^m}$$

8.6.7

$$Q_\nu^m(z) = (z^2-1)^{\frac{1}{2}m} \frac{d^m Q_\nu(z)}{dz^m},$$

$$Q_\nu^m(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m Q_\nu(x)}{dx^m}$$

$$\mu = \pm \frac{1}{2}$$

8.6.8

$$P_\nu^{\frac{1}{2}}(z) = (z^2-1)^{-1/4} (2\pi)^{-1/2} \{ [z + (z^2-1)^{1/2}]^{\nu+\frac{1}{2}} + [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \}$$

8.6.9

$$P_\nu^{-\frac{1}{2}}(z) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(z^2-1)^{-1/4}}{2\nu+1} \{ [z + (z^2-1)^{1/2}]^{\nu+\frac{1}{2}} - [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \}$$

8.6.10

$$Q_\nu^{\frac{1}{2}}(z) = i \left(\frac{1}{2}\pi\right)^{1/2} (z^2-1)^{-1/4} [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}$$

8.6.11

$$Q_\nu^{-\frac{1}{2}}(z) = -i (2\pi)^{1/2} \frac{(z^2-1)^{-1/4}}{2\nu+1} [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \quad *$$

8.6.12

$$P_\nu^{\frac{1}{2}}(\cos \theta) = \left(\frac{1}{2}\pi\right)^{-\frac{1}{2}} (\sin \theta)^{-\frac{1}{2}} \cos \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.13

$$Q_\nu^{\frac{1}{2}}(\cos \theta) = -\left(\frac{1}{2}\pi\right)^{\frac{1}{2}} (\sin \theta)^{-\frac{1}{2}} \sin \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.14

$$P_\nu^{-\frac{1}{2}}(\cos \theta) = \left(\frac{1}{2}\pi\right)^{-\frac{1}{2}} (\nu + \frac{1}{2})^{-1} (\sin \theta)^{-\frac{1}{2}} \sin \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.15

$$Q_\nu^{-\frac{1}{2}}(\cos \theta) = (2\pi)^{-\frac{1}{2}} (2\nu+1)^{-1} (\sin \theta)^{-\frac{1}{2}} \cos \left[(\nu + \frac{1}{2})\theta\right] \quad *$$

$$\mu = -\nu$$

8.6.16

$$P_\nu^{-\nu}(z) = \frac{2^{-\nu} (z^2-1)^{\frac{1}{2}\nu}}{\Gamma(\nu+1)}$$

8.6.17

$$P_\nu^{-\nu}(\cos \theta) = \frac{2^{-\nu} (\sin \theta)^\nu}{\Gamma(\nu+1)}$$

$$\mu = 0, \nu = n$$

(Rodrigues' Formula)

8.6.18

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n (z^2-1)^n}{dz^n}$$

$$8.6.19 \quad Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x)$$

where

$$W_{n-1}(x) = \frac{2n-1}{1 \cdot n} P_{n-1}(x) + \frac{2n-5}{3(n-1)} P_{n-3}(x) + \frac{2n-9}{5(n-2)} P_{n-5}(x) + \dots$$

$$= \sum_{m=1}^n \frac{1}{m} P_{m-1}(x) P_{n-m}(x)$$

$$W_{-1}(x) = 0$$

*See page 11.

$$\nu=0, 1$$

$$8.6.20 \quad \left[\frac{\partial P_\nu(\cos \theta)}{\partial \nu} \right]_{\nu=0} = 2 \ln (\cos \tfrac{1}{2}\theta)$$

$$8.6.21 \quad \left[\frac{\partial P_\nu^{-1}(\cos \theta)}{\partial \nu} \right]_{\nu=0} = -\tan \tfrac{1}{2}\theta - 2 \cot \tfrac{1}{2}\theta \ln (\cos \tfrac{1}{2}\theta)$$

$$8.6.22 \quad \left[\frac{\partial P_\nu^{-1}(\cos \theta)}{\partial \nu} \right]_{\nu=1} = -\tfrac{1}{2} \tan \tfrac{1}{2}\theta \sin^2 \tfrac{1}{2}\theta + \sin \theta \ln (\cos \tfrac{1}{2}\theta)$$

8.7. Trigonometric Expansions ($0 < \theta < \pi$)

$$8.7.1 \quad P_\nu^\mu(\cos \theta) = \pi^{-1/2} 2^{\mu+1} (\sin \theta)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_k (\nu+\mu+1)_k}{k! (\nu+\frac{3}{2})_k} \sin [(\nu+\mu+2k+1)\theta]$$

$$8.7.2 \quad Q_\nu^\mu(\cos \theta) = \pi^{1/2} 2^\mu (\sin \theta)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_k (\nu+\mu+1)_k}{k! (\nu+\frac{3}{2})_k} \cos [(\nu+\mu+2k+1)\theta]$$

$$8.7.3 \quad P_n(\cos \theta) = \frac{2^{2n+2} (n!)^2}{\pi (2n+1)!} \left[\sin (n+1)\theta + \frac{n+1}{2n+3} \sin (n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \sin (n+5)\theta + \dots \right]$$

$$8.7.4 \quad Q_n(\cos \theta) = \frac{2^{2n+1} (n!)^2}{(2n+1)!} \left[\cos (n+1)\theta + \frac{n+1}{2n+3} \cos (n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cos (n+5)\theta + \dots \right]$$

8.8. Integral Representations

(z not on the real axis between -1 and $-\infty$)

$$8.8.1 \quad P_\nu^\mu(z) = \frac{2^{-\nu} (z^2-1)^{-1/2}}{\Gamma(-\nu-\mu)\Gamma(\nu+1)} \int_0^\infty (z+\cosh t)^{\mu-\nu-1} (\sinh t)^{2\nu+1} dt \quad (\Re(-\mu) > \Re \nu > -1) \quad *$$

$$8.8.2 \quad Q_\nu^\mu(z) = \frac{e^{i\mu\pi} \sqrt{\pi} 2^{-\mu}}{\Gamma(\mu+\frac{1}{2})} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} (z^2-1)^{1/2} \int_0^\infty [z+(z^2-1)^{1/2} \cosh t]^{-\nu-\mu-1} (\sinh t)^{2\mu} dt \quad (\Re(\nu \pm \mu + 1) > 0) \quad *$$

$$8.8.3 \quad Q_n(z) = \frac{1}{2} \int_{-1}^1 (z-t)^{-1} P_n(t) dt = (-1)^{n+1} Q_n(-z)$$

(For other integral representations see [8.2].)

8.9. Summation Formulas

$$8.9.1 \quad (\xi-z) \sum_{m=0}^n (2m+1) P_m(z) P_m(\xi) = (n+1) [P_{n+1}(\xi) P_n(z) - P_n(\xi) P_{n+1}(z)]$$

$$8.9.2 \quad (\xi-z) \sum_{m=0}^n (2m+1) P_m(z) Q_m(\xi) = 1 - (n+1) [P_{n+1}(z) Q_n(\xi) - P_n(z) Q_{n+1}(\xi)]$$

8.10. Asymptotic Expansions

For fixed z and ν and $\Re \mu \rightarrow \infty$, 8.10.1-8.10.3 are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$. (Upper or lower signs according as $\Im z \gtrless 0$.)

$$8.10.1 \quad P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}{\pi\Gamma(\mu+1)} \left(\frac{z+1}{z-1}\right)^{1/2} \sin \mu\pi \left[F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}+\tfrac{1}{2}z) - \frac{\sin \nu\pi}{\sin \mu\pi} e^{\mp i\mu\pi} \left(\frac{z-1}{z+1}\right)^\mu F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}-\tfrac{1}{2}z) \right]$$

$$8.10.2 \quad Q_\nu^\mu(z) = \tfrac{1}{2} e^{i\mu\pi} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \left(\frac{z+1}{z-1}\right)^{1/2} \Gamma(\mu-\nu) \left[F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}+\tfrac{1}{2}z) - e^{\mp i\nu\pi} \left(\frac{z-1}{z+1}\right)^\mu F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}-\tfrac{1}{2}z) \right]$$

*See page II.

$$8.10.3 \quad Q_{\nu}^{-\mu}(z) = \frac{e^{-i\mu\pi} \csc[\pi(\nu-\mu)]}{2\pi\Gamma(1+\mu)} \left[e^{\mp i\nu\pi} \left(\frac{z+1}{z-1}\right)^{-i\mu} F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}-\tfrac{1}{2}z) \right. \\ \left. - \left(\frac{z-1}{z+1}\right)^{-i\mu} F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2}+\tfrac{1}{2}z) \right]$$

With μ replaced by $-\mu$, 8.1.2 is an asymptotic expansion for $P_{\nu}^{-\mu}(z)$ for fixed z and ν and $\mathcal{R} \mu \rightarrow \infty$ if z is not on the real axis between $-\infty$ and -1 .

For fixed z and μ and $\mathcal{R} \nu \rightarrow \infty$, 8.10.4 and 8.10.6 are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$; 8.10.5 if z is not on the real axis between $-\infty$ and $+1$.

$$8.10.4 \quad P_{\nu}^{\mu}(z) = (2\pi)^{-i} (z^2-1)^{-1/4} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \left\{ [z+(z^2-1)^{1/2}]^{\nu+1/2} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \tfrac{3}{2}+\nu; \frac{z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}) \right. \\ \left. + i e^{-i\mu\pi} [z-(z^2-1)^{1/2}]^{\nu+1/2} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \tfrac{3}{2}+\nu; \frac{-z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}) \right\}$$

$$8.10.5 \quad Q_{\nu}^{\mu}(z) = e^{i\mu\pi} (\tfrac{1}{2}\pi)^{1/2} (z^2-1)^{-1/4} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} [z-(z^2-1)^{1/2}]^{\nu+1/2} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \tfrac{3}{2}+\nu; \frac{-z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}})$$

$$8.10.6 \quad Q_{\nu}^{-\mu}(z) = \frac{e^{i\mu\pi} (\tfrac{1}{2}\pi)^{1/2} (z^2-1)^{-1/4}}{\sin[\pi(\mu-\nu)]} \frac{\Gamma(\mu+\nu)}{\Gamma(\tfrac{1}{2}-\mu)} \left\{ \cos \nu\pi [z+(z^2-1)^{1/2}]^{\nu-1/2} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \tfrac{1}{2}+\nu; \frac{z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}) \right. \\ \left. + i e^{i\nu\pi} \cos \mu\pi [z-(z^2-1)^{1/2}]^{\nu-1/2} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \tfrac{1}{2}+\nu; \frac{-z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}) \right\}$$

The related asymptotic expansion for $P_{\nu}^{\mu}(z)$ may be derived from 8.10.4 together with 8.2.1.

$$8.10.7 \quad P_{\nu}^{\mu}(\cos \theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\tfrac{1}{2}\pi \sin \theta)^{-i} \cos[(\nu+\tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{\mu\pi}{2}] + O(\nu^{-1})$$

$$8.10.8 \quad Q_{\nu}^{\mu}(\cos \theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \left(\frac{\pi}{2 \sin \theta}\right)^{1/2} \cos[(\nu+\tfrac{1}{2})\theta + \tfrac{\pi}{4} + \tfrac{\mu\pi}{2}] + O(\nu^{-1}) \quad (\epsilon < \theta < \pi - \epsilon, \epsilon > 0)$$

For other asymptotic expansions, see [8.7] and [8.9].

8.11. Toroidal Functions (or Ring Functions)

(Only special properties are given; other properties and representations follow from the earlier sections.)

$$8.11.1 \quad P_{\nu-1/2}^{\mu}(\cosh \eta) = [\Gamma(1-\mu)]^{-1} 2^{2\mu} (1-e^{-2\eta})^{-\mu} e^{-(\nu+1/2)\eta} F(\tfrac{1}{2}-\mu, \tfrac{1}{2}+\nu-\mu; 1-2\mu; 1-e^{-2\eta})$$

$$8.11.2 \quad P_{n-1/2}^m(\cosh \eta) = \frac{\Gamma(n+m+\frac{1}{2})(\sinh \eta)^m}{\Gamma(n-m+\frac{1}{2})2^m \sqrt{\pi} \Gamma(m+\frac{1}{2})} \int_0^{\pi} \frac{(\sin \varphi)^{2m} d\varphi}{(\cosh \eta + \cos \varphi \sinh \eta)^{n+m+1/2}}$$

$$8.11.3 \quad Q_{\nu-1/2}^{\mu}(\cosh \eta) = [\Gamma(1+\nu)]^{-1} \sqrt{\pi} e^{i\mu\pi} \Gamma(\tfrac{1}{2}+\nu+\mu) (1-e^{-2\eta})^{\mu} e^{-(\nu+1/2)\eta} F(\tfrac{1}{2}+\mu, \tfrac{1}{2}+\nu+\mu; 1+\nu; e^{-2\eta}) \quad *$$

$$8.11.4 \quad Q_{n-1/2}^m(\cosh \eta) = \frac{(-1)^m \Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{\infty} \frac{\cosh mt \, dt}{(\cosh \eta + \cosh t \sinh \eta)^{n+1/2}} \quad * \quad (n > m)$$

*See page 11.

8.12. Conical Functions

$$(P_{-\frac{1}{2}+i\lambda}(\cos \theta), Q_{-\frac{1}{2}+i\lambda}(\cos \theta))$$

(Only special properties are given as other properties and representations follow from earlier sections with $\nu = -\frac{1}{2} + i\lambda$ (λ , a real parameter) and $z = \cos \theta$.)

8.12.1

$$P_{-\frac{1}{2}+i\lambda}(\cos \theta) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\theta}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 4^2} \sin^4 \frac{\theta}{2} + \dots \quad (0 \leq \theta < \pi)$$

$$8.12.2 \quad P_{-\frac{1}{2}+i\lambda}(\cos \theta) = P_{-\frac{1}{2}-i\lambda}(\cos \theta)$$

$$8.12.3 \quad P_{-\frac{1}{2}+i\lambda}(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cosh \lambda t dt}{\sqrt{2}(\cos t - \cos \theta)}$$

8.12.4

$$Q_{-\frac{1}{2}+i\lambda}(\cos \theta) = \pm i \sinh \lambda \pi \int_0^\infty \frac{\cos \lambda t dt}{\sqrt{2}(\cosh t + \cos \theta)} + \int_0^\infty \frac{\cosh \lambda t dt}{\sqrt{2}(\cosh t - \cos \theta)}$$

8.12.5

$$P_{-\frac{1}{2}+i\lambda}(-\cos \theta) = \frac{\cosh \lambda \pi}{\pi} [Q_{-\frac{1}{2}+i\lambda}(\cos \theta) + Q_{-\frac{1}{2}-i\lambda}(\cos \theta)]$$

 * 8.13. Relation to Elliptic Integrals
(see chapter 17) ($\Re \eta > 0$)

$$8.13.1 \quad P_{-\frac{1}{2}}(z) = \frac{2}{\pi} \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{z-1}{z+1}}\right)$$

$$8.13.2 \quad P_{-\frac{1}{2}}(\cosh \eta) = \left[\frac{\pi}{2} \cosh \frac{\eta}{2}\right]^{-1} K\left(\tanh \frac{\eta}{2}\right)$$

$$8.13.3 \quad Q_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{2}{z+1}}\right)$$

$$8.13.4 \quad Q_{-\frac{1}{2}}(\cosh \eta) = 2e^{-\eta/2} K(e^{-\eta})$$

8.13.5

$$P_{\frac{1}{2}}(z) = \frac{2}{\pi} (z + \sqrt{z^2 - 1})^{1/2} E\left(\sqrt{\frac{2(z^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}}}\right)$$

$$8.13.6 \quad P_{\frac{1}{2}}(\cosh \eta) = \frac{2}{\pi} e^{\eta/2} E(\sqrt{1 - e^{-2\eta}})$$

8.13.7

$$Q_{\frac{1}{2}}(z) = z \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{2}{z+1}}\right) - [2(z+1)]^{1/2} E\left(\sqrt{\frac{2}{z+1}}\right) \quad (-1 < x < 1) *$$

$$8.13.8 \quad P_{-\frac{1}{2}}(x) = \frac{2}{\pi} K\left(\sqrt{\frac{1-x}{2}}\right)$$

$$8.13.9 \quad P_{-\frac{1}{2}}(\cos \theta) = \frac{2}{\pi} K\left(\sin \frac{\theta}{2}\right)$$

$$8.13.10 \quad Q_{-\frac{1}{2}}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) *$$

$$8.13.11 \quad P_{\frac{1}{2}}(x) = \frac{2}{\pi} \left[2E\left(\sqrt{\frac{1-x}{2}}\right) - K\left(\sqrt{\frac{1-x}{2}}\right) \right]$$

$$8.13.12 \quad Q_{\frac{1}{2}}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) - 2E\left(\sqrt{\frac{1+x}{2}}\right) *$$

8.14. Integrals

$$8.14.1 \quad \int_1^\infty P_\nu(x) Q_\nu(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} \quad (\Re \rho > \Re \nu > 0)$$

$$8.14.2 \quad \int_1^\infty Q_\nu(x) Q_\rho(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} [\psi(\rho + 1) - \psi(\nu + 1)] \quad (\Re(\rho + \nu) > -1, \rho + \nu + 1 \neq 0; \nu, \rho \neq -1, -2, -3, \dots)$$

$$8.14.3 \quad \int_1^\infty [Q_\nu(x)]^2 dx = (2\nu + 1)^{-1} \psi'(\nu + 1) \quad (\Re \nu > -\frac{1}{2})$$

$$8.14.4 \quad \int_{-1}^1 P_\nu(x) P_\rho(x) dx = \frac{2}{\pi^2} [(\rho - \nu)(\rho + \nu + 1)]^{-1} \{ 2 \sin \pi \nu \sin \pi \rho [\psi(\nu + 1) - \psi(\rho + 1)] + \pi \sin(\pi \rho - \pi \nu) \} \quad (\rho + \nu + 1 \neq 0)$$

$$8.14.5 \quad \int_{-1}^1 [P_\nu(x)]^2 dx = \frac{\pi^2 - 2(\sin \pi \nu)^2 \psi'(\nu + 1)}{\pi^2(\nu + \frac{1}{2})} *$$

$$8.14.6 \quad \int_{-1}^1 Q_\nu(x) Q_\rho(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} \{ [\psi(\nu + 1) - \psi(\rho + 1)] [1 + \cos \rho \pi \cos \nu \pi] - \frac{1}{2} \pi \sin(\nu \pi - \rho \pi) \} \quad (\rho + \nu + 1 \neq 0; \nu, \rho \neq -1, -2, -3, \dots)$$

$$8.14.7 \quad \int_{-1}^1 [Q_\nu(x)]^2 dx = (2\nu + 1)^{-1} \{ \frac{1}{2} \pi^2 - \psi'(\nu + 1) [1 + (\cos \nu \pi)^2] \} \quad (\nu \neq -1, -2, -3, \dots)$$

$$8.14.8 \quad \int_{-1}^1 P_\nu(x) Q_\rho(x) dx = [(\nu - \rho)(\rho + \nu + 1)]^{-1} \left\{ 1 - \cos(\rho\pi - \nu\pi) - \frac{2}{\pi} \sin \pi\nu \cos \pi\nu [\psi(\nu+1) - \psi(\rho+1)] \right\} \\ (\Re \nu > 0, \Re \rho > 0, \rho \neq \nu)$$

$$8.14.9 \quad \int_{-1}^1 P_\nu(x) Q_\nu(x) dx = -\frac{1}{\pi} (2\nu+1)^{-1} \sin 2\nu\pi \psi'(\nu+1) \quad (\Re \nu > 0) \\ (m, n, l \text{ positive integers})$$

8.14.10

$$\int_{-1}^1 Q_n^m(x) P_l^m(x) dx = (-1)^m \frac{1 - (-1)^{l+n} (n+m)!}{(l-n)(l+n+1)(n-m)!}$$

$$8.14.11 \quad \int_{-1}^1 P_n^m(x) P_l^m(x) dx = 0 \quad (l \neq n)$$

$$8.14.12 \quad \int_{-1}^1 P_n^m(x) P_l^m(x) (1-x^2)^{-1} dx = 0 \quad (l \neq m)$$

$$8.14.13 \quad \int_{-1}^1 [P_n^m(x)]^2 dx = (n + \frac{1}{2})^{-1} (n+m)! / (n-m)!$$

8.14.14

$$\int_{-1}^1 (1-x^2)^{-1} [P_n^m(x)]^2 dx = (n+m)! / m(n-m)!$$

8.14.15

$$\int_0^1 P_\nu(x) x^\rho dx = \frac{\pi^{1/2} 2^{-\rho-1} \Gamma(1+\rho)}{\Gamma(1+\frac{1}{2}\rho - \frac{1}{2}\nu) \Gamma(\frac{1}{2}\rho + \frac{1}{2}\nu + \frac{3}{2})} \quad (\Re \rho > -1)$$

8.14.16

$$\int_0^\pi (\sin t)^{\alpha-1} P_\nu^{-\mu}(\cos t) dt = \frac{2^{-\mu} \pi \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\mu)}{\Gamma(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\nu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\nu) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})} \quad (\Re(\alpha \pm \mu) > 0)$$

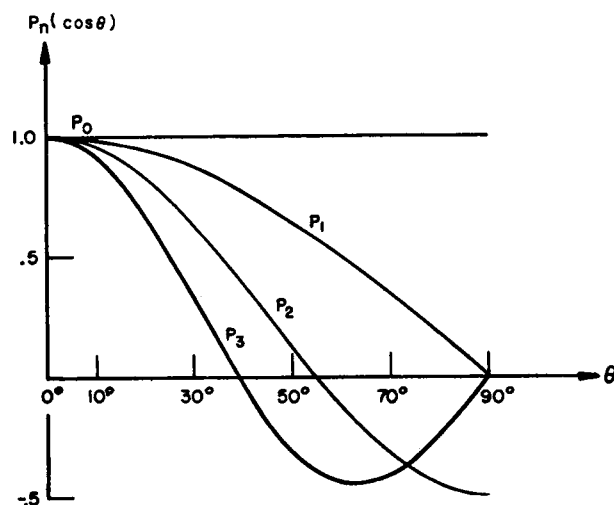
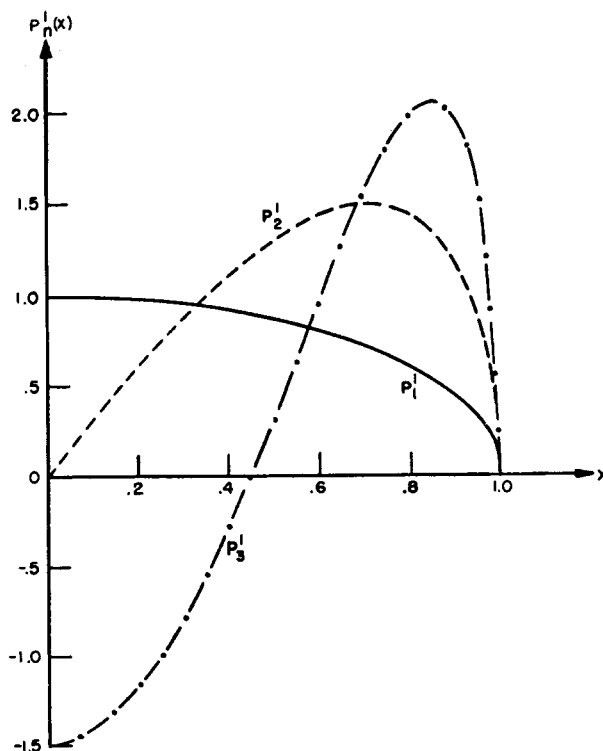
8.14.17

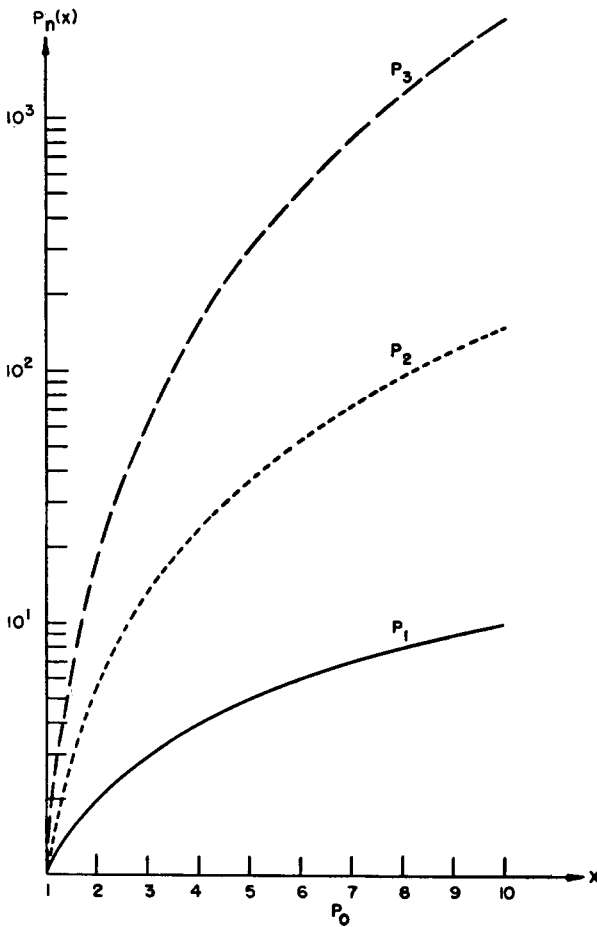
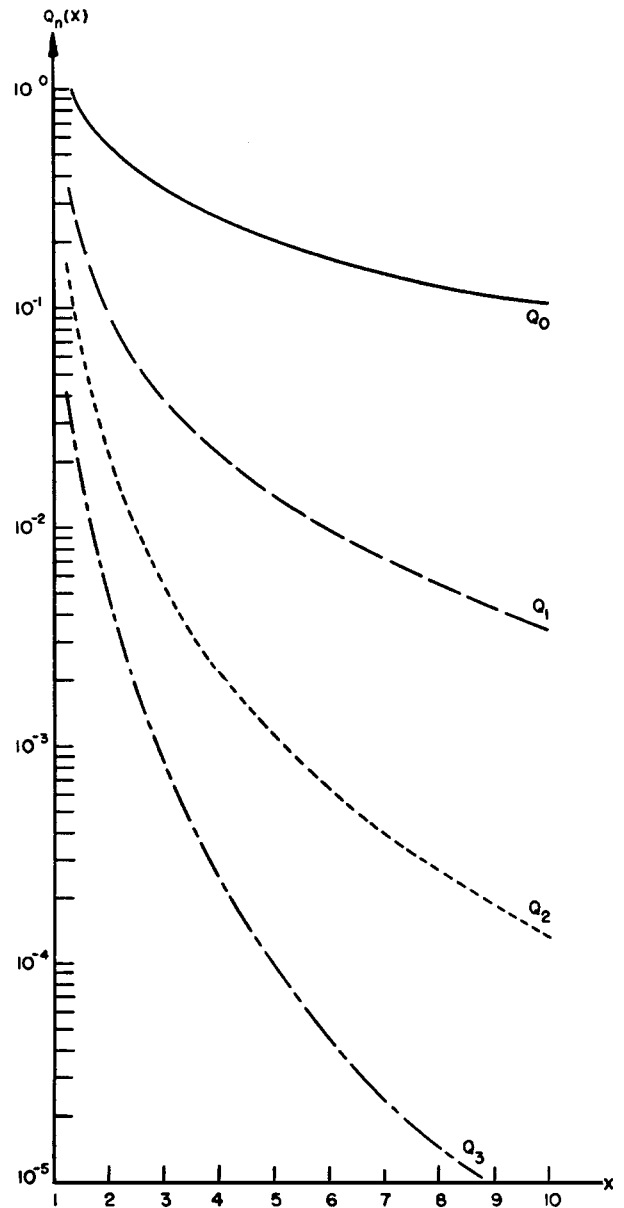
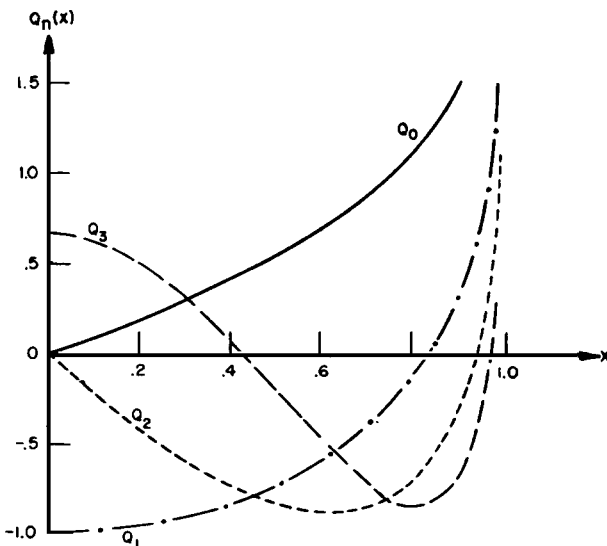
$$P_\nu^{-m}(z) = (z^2 - 1)^{-1/2 m} \int_1^z \cdots \int_1^z P_\nu(z) (dz)^m$$

8.14.18

$$Q_\nu^{-m}(z) = (-1)^m (z^2 - 1)^{-1/2 m} \int_z^\infty \cdots \int_z^\infty Q_\nu(z) (dz)^m$$

For other integrals, see [8.2], [8.4] and chapter 22.

FIGURE 8.1. $P_n(\cos \theta)$. $n=0(1)3$.FIGURE 8.2. $P_n^l(x)$. $n=1(1)3, x \leq 1$.


 FIGURE 8.3. $P_n(x)$. $n=0(1)3$, $x \geq 1$.

 FIGURE 8.5. $Q_n(x)$. $n=0(1)3$, $x > 1$.

 FIGURE 8.4. $Q_n(x)$. $n=0(1)3$, $x < 1$.

Numerical Methods

8.15. Use and Extension of the Tables

Computation of $P_n(x)$

For all values of x there is very little loss of significant figures (except at zeros) in using the recurrence relation 8.5.3 for increasing values of n .

Example 1. Compute $P_n(x)$ for $x=.31415\ 92654$ and $x=2.6$ for $n=2(1)8$.

n	$P_n(.31415\ 92654)$	$P_n(2.6)$
0	1	1
1	.31415 92654	2.6
2	-.35195 59340	9.64
3	-.39372 32064	40.04
4	.04750 63122	174.952
5	.34184 27517	786.74336
6	.15729 86975	3604.350016
7	-.20123 39354	16729.51005
8	-.25617 29328	78402.55522

Computing $P_8(x)$ using **Table 22.9** carrying ten significant figures, $P_8(.31415\ 92654) = -.25617\ 2933$ and $P_8(2.6) = 78402.55526$.

Computation of $Q_n(x)$

For $x < 1$, use of 8.5.3 for increasing values of n leads to very little loss of significant figures. However, for $x > 1$, the recurrence relation 8.5.3 should be used only for decreasing values of n , after having first obtained Q_n using the formulas in terms of hypergeometric functions.

Example 2. Compute $Q_n(x)$ for $x = .31415\ 92654$ and $n = 0(1)4$.

With the aid of 8.4.2 and 8.4.4 we obtain

n	$Q_n(.31415\ 92654)$
0	.32515 34813
1	-.89785 00212
2	-.58567 85953
3	.29190 60854
4	.59974 26989

Using the results of **Example 1** together with 8.6.19, we find $Q_4(x) = \frac{1}{2}P_4(x)\ln\left(\frac{1+x}{1-x}\right) - W_3(x)$ where $W_3 = \frac{7}{4}P_3 + \frac{1}{3}P_1$, giving $Q_4(.31415\ 92654) = .59974\ 26989$.

Example 3. Compute $Q_5(x)$ for $x = 2.6$.

Ten terms in the series for $F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \nu+\frac{3}{2}, \frac{1}{2^2}\right)$ of 8.1.3 are necessary to obtain nine significant figures giving $Q_5(2.6) = 4.8182\ 4468 \times 10^{-5}$. Using 8.5.3 with increasing values of n carrying ten significant figures we obtain

n	$Q_n(2.6)$
0	.40546 51081
1	.05420 928
2	.00868 364
3	.00148 95
4	.00026 49
5	.00004 81

where Q_0 and Q_1 are obtained using 8.4.2 and 8.4.4.

Computation of $P_{\pm\frac{1}{2}}(x)$, $Q_{\pm\frac{1}{2}}(x)$

For all values of x , $P_{\pm\frac{1}{2}}(x)$ and $Q_{\pm\frac{1}{2}}(x)$ are most easily computed by means of 8.13.

Example 4. Compute $Q_{-\frac{1}{2}}(x)$ for $x = 2.6$.

Using 8.13.3 and interpolating in **Table 17.1** for $K(.5)$, we find

$$\begin{aligned} Q_{-\frac{1}{2}}(2.6) &= \sqrt{\frac{2}{x+1}} K\left(\sqrt{\frac{2}{x+1}}\right) \\ &= (.74535\ 59925)(1.90424\ 1417) \\ &= 1.41933\ 7751. \end{aligned}$$

On the other hand, at least nine terms in the expansion of $F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \nu+\frac{3}{2}, \frac{1}{2^2}\right)$ of 8.1.3 are necessary to obtain comparable accuracy.

References

- Texts**
- [8.1] A. Erdélyi et al., Higher transcendental functions, vol. 1, ch. 3 (McGraw-Hill Book Co., Inc., New York, N.Y., 1953).
 - [8.2] E. W. Hobson, The theory of spherical and ellipsoidal harmonics (Chelsea Publishing Co., New York, N.Y., 1955).
 - [8.3] J. Lense, Kugelfunktionen (Akademische Verlagsgesellschaft, Leipzig, Germany, 1950).
 - [8.4] T. M. MacRobert, Spherical harmonics, 2d rev. ed. (Dover Publications, Inc., New York, N.Y., 1948).
 - [8.5] W. Magnus and F. Oberhettinger, Formulas and theorems for the special functions of mathematical physics (Chelsea Publishing Co., New York, N.Y., 1949).
 - [8.6] G. Prasad, A treatise on spherical harmonics and the functions of Bessel and Lamé, Part II (Advanced) (Mahamandal Press, Benares City, India, 1932).
- Tables**
- [8.7] L. Robin, Fonctions sphériques de Legendre e fonctions sphéroïdales. Tome I, II, III (Gauthier-Villars, Paris, France, 1957).
 - [8.8] C. Snow, Hypergeometric and Legendre functions with applications to integral equations of potential theory, NBS Applied Math. Series 19 (U.S. Government Printing Office, Washington, D.C., 1952).
 - [8.9] R. C. Thorne, The asymptotic expansion of Legendre functions of large degree and order, Philos. Trans. Roy. Soc. London **249**, 597-620 (1957).
 - [8.10] H. Bateman, Some problems in potential theory, Mess. of Math., **617**, 52, 73-75 (1922). $P_n(\cosh \sigma)$, $Q_n(\cosh \sigma)$, $P'_n(\cosh \sigma)$, $Q'_n(\cosh \sigma)$; $\cosh \sigma = 1.1$, $n = 0(1)20, 10D$; $\cosh \sigma = 1.2, 2, 3$; $n = 0(1)10$, exact or 10D.

- [8.11] Centre National d'Études des Télécommunications, Tables des fonctions de Legendre associées. Fonction associée de première espèce $P_n^m(\cos \theta)$ (Éditions de La Revue d'Optique, Paris, France, 1952). $n = -\frac{1}{2}(1)10$, $m = 0(1)5$, $\theta = 0^\circ(1^\circ)90^\circ$ (variable number of figures).
- [8.12] Centre National d'Études des Télécommunications, Tables numérique des fonctions associées de Legendre. Fonctions associées de première espèce $P_n^m(\cos \theta)$ (Éditions de La Revue d'Optique, Paris, France, 1959). $n = -\frac{1}{2}(1)10$, $m = 0(1)2$, $\theta = 0^\circ(1^\circ)180^\circ$ (variable number of figures).
- [8.13] G. C. Clark and S. W. Churchill, Table of Legendre polynomials $P_n(\cos \theta)$ for $n = 0(1)80$ and $\theta = 0^\circ(1^\circ)180^\circ$, Engineering Research Institute Publications (Univ. of Michigan Press, Ann Arbor, Mich., 1957).
- [8.14] R. O. Gumprecht and G. M. Sliepcevic, Tables of functions of the first and second partial derivatives of Legendre polynomials (Univ. of Michigan Press, Ann Arbor, Mich., 1951). Values of $[x\pi_n - (1-x^2)\pi_n'] \cdot 10^4$ and $\pi_n 10^4$ for $\gamma = 0^\circ(10^\circ)170^\circ(1^\circ)180^\circ$, $n = 1(1)420$, 5S.
- [8.15] M. E. Lynam, Table of Legendre functions for complex arguments TG-323, The Johns Hopkins Univ. Applied Physics Laboratory, Baltimore, Md. (1958).
- [8.16] National Bureau of Standards, Tables of associated Legendre functions (Columbia Univ. Press, New York, N.Y., 1945). $P_n^m(\cos \theta)$, $\frac{d}{d\theta} P_n^m(\cos \theta)$, $n = 1(1)10$, $m(\leq n) = 0(1)4$, $\theta = 0^\circ(1^\circ)90^\circ$, 6S;
- $P_n^m(x)$, $\frac{d}{dx} P_n^m(x)$, $n = 1(1)10$, $(-1)^m Q_n^m(x)$, $(-1)^{m+1} \frac{d}{dx} Q_n^m(x)$, $n = 0(1)10$, $m(\leq n) = 0(1)4$, $x = 1(1)10$, 6S or exact; $i^{-n} P_n^m(ix)$, $i^{-n} \frac{d}{dx} P_n^m(ix)$, $n = 1(1)10$, $i^{n+2m+1} Q_n^m(ix)$, $i^{n+2m-1} \frac{d}{dx} Q_n^m(ix)$, $n = 0(1)10$, $m(\leq n) = 0(1)4$, $x = 0(1)10$, 6S; $P_{n+\frac{1}{2}}^m(x)$, $\frac{d}{dx} P_{n+\frac{1}{2}}^m(x)$, $(-1)^m Q_{n+\frac{1}{2}}^m(x)$, $(-1)^{m+1} \frac{d}{dx} Q_{n+\frac{1}{2}}^m(x)$, $n = -1(1)4$, $m = 0(1)4$, $x = 1(1)10$, 4-6S.
- [8.17] G. Prevost, Tables des fonctions sphériques et de leurs intégrales (Gauthier-Villars, Bordeaux and Paris, France, 1933). $P_n(x)$, $\int_0^x P_n(t) dt$, $n = 1(1)10$; $P_n^i(x)$, $\int_0^x P_n^i(t) dt$, $n = 0(1)8$, $j = 0(1)n$, $x = 0(.01)1$, 5S.
- [8.18] H. Tallqvist, Sechstellige Tafeln der 32 ersten Kugelfunktionen $P_n(\cos \theta)$, Acta Soc. Sci. Fenn., Nova Series A, II, 11 (1938). $P_n(\cos \theta)$, $n = 1(1)32$, $0^\circ(10')90^\circ$; 6D.
- [8.19] H. Tallqvist, Acta Soc. Sci. Fenn., Nova Series A, II, 4(1937). $P_n(x)$, $n = 1(1)16$, $x = 0(.001)1$, 6D.
- [8.20] H. Tallqvist, Tafeln der Kugelfunktionen $P_{25}(\cos \theta)$ bis $P_{32}(\cos \theta)$, Soc. Sci. Fenn. Comment. Phys.-Math., VI, 10(1932). $P_n(\cos \theta)$, $n = 25(1)32$, $\theta = 0^\circ(1^\circ)90^\circ$, 5D.
- [8.21] H. Tallqvist, Tafeln der 24 ersten Kugelfunktionen $P_n(\cos \theta)$, Soc. Sci. Fenn. Comment. Phys.-Math., VI, 3(1932). $P_n(\cos \theta)$, $n = 1(1)24$, $\theta = 0^\circ(1^\circ)90^\circ$, 5D.